IOP Publishing Physica Scripta
Phys. Scr. 95 (2020) 105009 (10pp) https://doi.org/10.1088/1402-4896/abbaa3

Mathematical computations for Peristaltic flow of heated non-Newtonian fluid inside a sinusoidal elliptic duct

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Received 9 August 2020, revised 4 September 2020
Accepted for publication 22 September 2020
Published 30 September 2020

Abstract

This research work interprets the mathematical study of peristaltic flow of non-Newtonian fluid across an elliptical duct. The heat transfer mechanism for this elliptical duct problem is also considered in detail. The mathematical equations for Casson fluid model are developed and then by using appropriate transformations and long wavelength approximation, this mathematical problem is converted into its dimensionless form. After converting the problem in dimensionless form, we have obtained partial differential equations for both velocity and temperature profiles. These partial differential equations are solved subject to given boundary conditions over elliptical cross sections and exact mathematical solutions are obtained. The results are further discussed by plotting graphical results for velocity, pressure gradient, temperature, pressure rise and streamlines.

Keywords: peristaltic flow, elliptical duct, casson fluid, viscous dissipation

(Some figures may appear in colour only in the online journal)

Nomenclature

$(\bar{X}, \bar{Y}, \bar{Z})$	Cartesian coordinate system
$(\bar{U}, \bar{V}, \bar{W})$	Velocity components
a_0, b_0	Half-axes of ellipse
d	Wave amplitude
λ	Wavelength
c	Velocity of propagation
$\bar{T}_w$	Tube's wall temperature
D_h	Hydraulic diameter of ellipse
e	Eccentricity of ellipse
$\bar{T}_b$	Bulk temperature
ϕ	Occlusion
δ	Aspect ratio
B_r	Brickmann number
β	Casson parameter
μ_B	Plastic viscosity
p_y	Yield Stress

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1. Introduction

The phenomenon of fluid propulsion across a channel with deformable walls is known as peristalsis. The walls of such flexible tubes contract and relax in such a systematic pattern along the axial coordinate that the fluid also moves toward the axial coordinate of the tube. The immense number of peristalsis applications in the area of biology, engineering, biomedicine and industrial problems has made it a very important topic for researchers. The mechanical devices that operate on peristalsis mechanisms are used to transport food, blood, slurry and corrosive liquids [1]. Its industrial applications include processing of food, treatment of water and agriculture etc. Barton [2] had mathematically examined the peristaltic flow across the tubes having much larger peristaltic wavelength when compared to the average radius of the tube. Bohme *et al* [3] had provided the study of non-Newtonian flow across ducts having sinusoidally deforming walls. In their work, a constant pressure is considered over cross-section. The asymmetrical channel peristaltic flow of a non-Newtonian Casson fluid with applications in refinement of crude oil was mathematically interpreted by Akbar [4]. Further, some more fruitful articles on flow of non-Newtonian fluids across channels with sinusoidally deformable walls is provided [5–13].

Recently, the flow inside elliptical ducts has also achieved the interest of many researchers because of their huge applications. Abdel Wahid *et al* [14] had experimentally conducted the study of heat transfer and laminar flow across a duct with elliptical cross section. The heat transfer phenomenon with viscous dissipation effects inside a duct having elliptical cross section was studied by Ragueb *et al* [15]. Maia *et al* [16] had investigated the fully developed flow with constant properties inside a duct having elliptical cross-section. There are many research articles in the literature that mathematically addresses the peristaltic flow inside different geometries like cylinders, asymmetric channels and rectangular ducts. But the study of peristaltic flow inside elliptical ducts is not much explored mathematically despite its immense industrial and engineering applications. This huge gap in literature needs to be filled, because the peristaltic flow in an elliptical duct also has its worth like the study of peristaltic flow inside cylindrical and rectangular ducts. An advantage of using the elliptic duct than circular duct is that the heat transfer coefficient is increased. A reduction of drag force is also expected in case of elliptical duct as compared to circular duct. Further, elliptic duct has a less pressure drop than circular one. Some recent articles related to peristaltic flow problems are [17–38].

This research article addresses the mathematical study of peristaltic flow of any non-Newtonian fluid within a duct having elliptical cross section for first the time. The non-Newtonian fluid considered in this study is the Casson fluid non-Newtonian model. The heat transfer mechanism for this elliptical duct problem is also studied in detail. Energy equation involves the viscous dissipation term and it provides a full analysis of heat transfer phenomenon. The mathematical equations for Casson fluid model are developed and then by using appropriate transformations, we have converted this mathematical problem into its dimensionless form. After converting the problem in dimensionless form, we have obtained partial differential equations for both velocity and temperature profile equations. These partial differential equations are solved subject to given boundary conditions over elliptical cross sections and exact mathematical solutions are obtained. The results are further discussed by plotting graphical results for velocity, pressure gradient, temperature and pressure rise. The graphical outcomes include both 2-dimensional and 3-dimensional velocity and temperature results.

2. Mathematical formulation

The incompressible, peristaltic flow of Casson fluid is studied across a duct having elliptical cross-section see figure 1. A Cartesian coordinate system $(\bar{X}, \bar{Y}, \bar{Z})$ is utilized for this mathematical study.

The following sinusoidal equations are provided to describe the geometry of deformable walls [39].

$$\bar{a}(\bar{Z}, \bar{t}) = a_0 + d\ \mathrm{Sin}\left(\frac{2\pi}{\lambda}(\bar{Z} - c\bar{t})\right),$$
$$\bar{b}(\bar{Z}, \bar{t}) = b_0 + d\ \mathrm{Sin}\left(\frac{2\pi}{\lambda}(\bar{Z} - c\bar{t})\right), \tag{1}$$

The dimensional form of equations governing an incompressible, non-Newtonian flow are given as

$$\frac{\partial \bar{U}}{\partial \bar{X}} + \frac{\partial \bar{V}}{\partial \bar{Y}} + \frac{\partial \bar{W}}{\partial \bar{Z}} = 0, \tag{2}$$

$$\rho\left(\frac{\partial \bar{U}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{U}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{U}}{\partial \bar{Y}} + \bar{W}\frac{\partial \bar{U}}{\partial \bar{Z}}\right) = -\frac{\partial \bar{P}}{\partial \bar{X}} + \frac{\partial \bar{S}_{\bar{X}\bar{X}}}{\partial \bar{X}} + \frac{\partial \bar{S}_{\bar{Y}\bar{X}}}{\partial \bar{Y}} + \frac{\partial \bar{S}_{\bar{Z}\bar{X}}}{\partial \bar{Z}}, \tag{3}$$

$$\rho\left(\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{V}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{V}}{\partial \bar{Y}} + \bar{W}\frac{\partial \bar{V}}{\partial \bar{Z}}\right) = -\frac{\partial \bar{P}}{\partial \bar{Y}} + \frac{\partial \bar{S}_{\bar{X}\bar{Y}}}{\partial \bar{X}} + \frac{\partial \bar{S}_{\bar{Y}\bar{Y}}}{\partial \bar{Y}} + \frac{\partial \bar{S}_{\bar{Z}\bar{Y}}}{\partial \bar{Z}}, \tag{4}$$

$$\rho\left(\frac{\partial \bar{W}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{W}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{W}}{\partial \bar{Y}} + \bar{W}\frac{\partial \bar{W}}{\partial \bar{Z}}\right) = -\frac{\partial \bar{P}}{\partial \bar{Z}} + \frac{\partial \bar{S}_{\bar{X}\bar{Z}}}{\partial \bar{X}} + \frac{\partial \bar{S}_{\bar{Y}\bar{Z}}}{\partial \bar{Y}} + \frac{\partial \bar{S}_{\bar{Z}\bar{Z}}}{\partial \bar{Z}}, \tag{5}$$

$$\rho C_p\left(\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{T}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{T}}{\partial \bar{Y}} + \bar{W}\frac{\partial \bar{T}}{\partial \bar{Z}}\right) = k\left(\frac{\partial^2 \bar{T}}{\partial \bar{X}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{Y}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{Z}^2}\right) + \bar{S}_{\bar{X}\bar{X}}\frac{\partial \bar{U}}{\partial \bar{X}} + \bar{S}_{\bar{X}\bar{Y}}\frac{\partial \bar{U}}{\partial \bar{Y}} + \bar{S}_{\bar{X}\bar{Z}}\frac{\partial \bar{U}}{\partial \bar{Z}} + \bar{S}_{\bar{Y}\bar{X}}\frac{\partial \bar{V}}{\partial \bar{X}} + \bar{S}_{\bar{Y}\bar{Y}}\frac{\partial \bar{V}}{\partial \bar{Y}} + \bar{S}_{\bar{Y}\bar{Z}}\frac{\partial \bar{V}}{\partial \bar{Z}} + \bar{S}_{\bar{Z}\bar{X}}\frac{\partial \bar{W}}{\partial \bar{X}} + \bar{S}_{\bar{Z}\bar{Y}}\frac{\partial \bar{W}}{\partial \bar{Y}} + \bar{S}_{\bar{Z}\bar{Z}}\frac{\partial \bar{W}}{\partial \bar{Z}}, \tag{6}$$

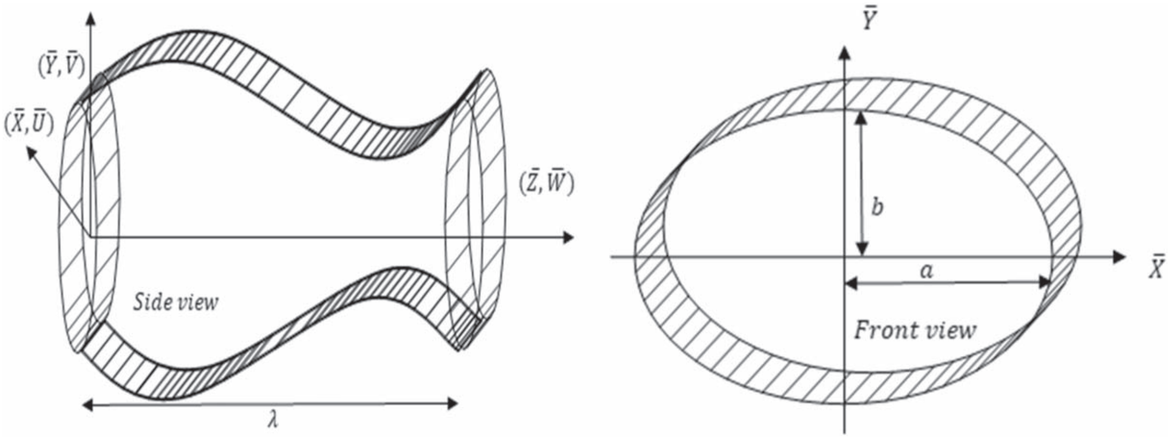

Figure 1. Geometry of Elliptical duct.

The dimensional form of associated boundary conditions are given as

$$\bar{W}=0,\ \bar{T}=\bar{T}_w,\ \text{for}\ \frac{\bar{x}^2}{\bar{a}^2}+\frac{\bar{y}^2}{\bar{b}^2}=1, \tag{7}$$

The given tensor [40] defines the extra stresses for Casson fluid as follows

$$\bar{S}_{ij}=\left(\mu_B+\frac{p_y}{\sqrt{2\pi_c}}\right)\bar{e}_{ij}, \tag{8}$$

Where

$$\bar{e}_{ij}=\frac{\partial\bar{V}_i}{\partial\bar{X}_j}+\frac{\partial\bar{V}_j}{\partial\bar{X}_i}, \tag{9}$$

The equations given below relate the two frames, (i.e. fixed and moving one).

$$\begin{aligned}&\bar{x}=\bar{X},\ \bar{y}=\bar{Y},\ \bar{z}=\bar{Z}-c\bar{t},\ \bar{p}=\bar{P},\\&\bar{u}=\bar{U},\ \bar{v}=\bar{V},\ \bar{w}=\bar{W}-c,\end{aligned} \tag{10}$$

The dimensionless variables are given as

$$\begin{aligned}&x=\frac{\bar{x}}{D_h},\ y=\frac{\bar{y}}{D_h},\ z=\frac{\bar{z}}{\lambda},\ t=\frac{c\bar{t}}{\lambda},\ w=\frac{\bar{w}}{c},\\&p=\frac{D_h^2\bar{p}}{\mu\lambda c},\ \theta=\frac{\bar{T}-\bar{T}_w}{\bar{T}_b-\bar{T}_w},\ \delta=\frac{b_0}{a_0},\ \phi=\frac{d}{b_0},\\&B_r=\frac{\mu c^2}{k(\bar{T}_b-\bar{T}_w)},\ u=\frac{\lambda\bar{u}}{D_h c},\ v=\frac{\lambda\bar{v}}{D_h c},\\&\beta=\frac{\mu_B\sqrt{2\pi_c}}{p_y},\ S_{ij}=\frac{D_h\bar{S}_{ij}}{c\mu_f},\ a=\frac{\bar{a}}{D_h},\ b=\frac{\bar{b}}{D_h},\end{aligned} \tag{11}$$

The hydraulic diameter D_h of ellipse is provided as

$$D_h=\frac{\pi b_0}{E(e)}, \tag{12}$$

Here $e=\sqrt{1-\delta^2}$, is ellipse eccentricity and $E(e)$ is elliptical integral of second kind [41].

$$E(e)=\int_0^{\pi/2}\sqrt{1-e^2\,\mathrm{Sin}^2\,\alpha}\,d\alpha, \tag{13}$$

Equations (10) and (11) are used in equations (2)–(6) and then applying lubrication approximation, (i.e. $\lambda\to\infty$), we have following non-dimensional governing equations

$$\frac{\partial p}{\partial x}=0, \tag{14}$$

$$\frac{\partial p}{\partial y}=0, \tag{15}$$

$$\frac{dp}{dz}=\left(1+\frac{1}{\beta}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right), \tag{16}$$

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+B_r\left(1+\frac{1}{\beta}\right)\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]=0, \tag{17}$$

With the non-dimensional boundary conditions

$$w=-1,\ \text{for}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{18}$$

$$\theta=0,\ \text{for}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{19}$$

where $a=\frac{E(e)}{\pi}\left[\frac{1}{\delta}+\phi\ \mathrm{Sin}(2\pi z)\right]$, and $b=\frac{E(e)}{\pi}[1+\phi\ \mathrm{Sin}(2\pi z)]$.

3. Exact solution

An exact mathematical solution is developed for velocity profile by solving the partial differential equation given in equation (16), with relevant boundary conditions provided in equation (18) and this obtained velocity solution satisfies both equation and boundary condition. The solution for both velocity and temperature profiles are obtained by using the method provided in the work of Hayman and Shanidze [42].

$$w(x,y)=-1+\frac{1}{2}\frac{dp}{dz}\frac{a^2b^2}{a^2+b^2}\left(\frac{\beta}{1+\beta}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right), \tag{20}$$