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A new analytic solution for buckling of doubly clamped nano-actuators with integro differential governing equation using Duan–Rach Adomian decomposition method

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ABSTRACT

The Duan–Rach modified Adomian Decomposition Method (ADM) is utilized to obtain a convergent series solution for buckling of nano-actuators subject to different nonlinear forces. To achieve this purpose, a general type of the governing equation for nano-actuators, including integro-differential terms and nonlinear forces, is considered. The adopted governing equation for the nano-actuators is a non-linear fourth-order integro-differential boundary value equation. A new fast convergent parameter, Duan's parameter, is applied to accelerate the convergence rate of the solution. The obtained solution is an explicit polynomial series solution free of any undetermined coefficient. Thus, it can facilitate the design of nano-actuators. As a case study, the results of the approach are compared with the results of Wazwaz Modified Adomian Decomposition Method (MADM) (with undetermined coefficients) as well as a numerical method. The results indicate the remarkable robustness of the present approach.

1. Introduction

A nano/micro actuator is a basic inevitable part of many nano/micro electro mechanical systems. In nano/micro mechanical sensors or actuators, the actuator consists of a beam suspended over a substrate. Applying a voltage difference between the beam and the substrate induces electrostatic field, and as a result, the electrostatic forces attracts the beam into the substrate. The system (i.e. beam and substrate) acts as a capacitor, in which the motion of the beam can be detected by the capacitive change as a signal [1]. The nano-actuators are subject to different inherently nonlinear forces including van der Waals attractions, Casimir force, dielectric effects and fringing field effects [2–4]. The axial forces in the clamped-clamped type of nano-actuators play a crucial role and should be taken into account. The presence of such axial forces adds an integro-differential term to the governing equation of the nano-actuators [2,5–7]. In most of the applications of nano-/micro-actuators, the actuators are manufactured in packs as many as thousands for sensors and billions for chipsets. Hence, developing accurate and fast methods for analysis of nano/micro structures consisting of these actuators is highly demanded.

In many of the recent studies, the modified Adomian Decomposition Method (ADM) with undetermined coefficients were utilized has been utilized to obtain an analytical solution for buckling of the nano-actuators [3,4,8–14]. The results of these

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studies indicate the robustness and capability of ADM to deal with the nonlinearity of the governing equations and provide acceptable accuracy. In the method of undetermined coefficients utilized in [3,4,8–14], there are undetermined coefficients in the solution, which should be determined later using the boundary conditions. Thus, the obtained solution is not in a fully explicit form, and hence, the relations between the affective parameters are not clear. In addition, in many cases, the algebraic equations for evaluating the undetermined coefficients are highly nonlinear and demands numerical procedure. Moreover, in most cases, there are several sets of roots for undetermined coefficients, in which finding and choosing the physical set of roots is a challenge. These are the main disadvantages of ADM which limits the practical application.

Recently, Duan and Rach [15] proposed a new modification of Adomian Decomposition Method (ADM) to deal with the boundary-value problems. They came with the idea that they could use all of the boundary conditions to build a decomposed solution free of any undetermined coefficient. Then, a recursive scheme can be utilized to evaluate the solution in different stages. The method of Duan and Rach is capable of obtaining a fully explicit solution, which is free of any undetermined coefficient. Duan et al. [16] embedded a fast convergent parameter, which can remarkably accelerate the convergence rate of the solution.

In the present study, a general form of the governing equation of nano-actuators, including the integro-differential term is considered. The Duan–Rach modified Adomian decomposition method is utilized to overcome the shortcomings of the conventional Adomian decomposition methods and obtain an explicit analytical solution for buckling of nano-actuators in the presence of different nonlinear effects.

2. Mathematical model

A general form of the governing equation of a nano-actuator beam, including the effect of axial loads and different types of nonlinear forces, in a non-dimensional form can be written as:

$$\frac{d^4u}{dx^4} - \left(\eta \int_0^1 \left(\frac{du}{dx}\right)^2 dx + P\right)\frac{d^2u}{dx^2} = -\frac{\alpha}{u^\zeta} - \frac{\beta}{(\kappa+u)^2} - \frac{\gamma}{u}, \tag{1}$$

where u is the non-dimensional deflection of the beam (dependent variable) and x is the non-dimensional length of the beam (independent variable). P, η, β, κ, γ and α are non-dimensional parameters and ζ is an integer positive number. P and η represent the effect of axial forces, β denotes the effect of the external applied voltage, κ represents the effect of a dielectric layer, and γ represents the fringing field or the capillary effect. In the case of $\zeta = 3$, α denotes the van der Waals effects, and in the case of $\zeta = 4$, α denotes the Casimir effect. It is worth noticing that in the previous studies, some of the non-dimensional parameters of Eq. (1) were neglected. Here, in Eq. (1), we considered all of the terms, which are utilized in different studies, to form a general differential equations. For example, the terms of P in [6], $\eta \int_0^1 (\frac{du}{dx})^2 dx$ in [2,6,7], β in [17,18], γ in [12,13,18], α/u^3 (i.e. $\zeta = 3$) in [3,12], α/u^4 (i.e. $\zeta = 4$) in [4,10,13] and κ in [2] have been utilized. Neglecting some of these terms reduces the present governing equation to each of the mentioned studies. The clamped-clamped nano-actuators are subject to the following boundary conditions [2,6]:

$$u(0) = 1, \quad u'(0) = 0, \quad u(1) = 1, \quad u'(1) = 0. \tag{2}$$

Therefore the goal of the present study is utilizing and modifying the Duan–Rach Adomian method to obtain an explicit convergent series solution for Eq. (1) subject to boundary conditions of Eq. (2).

3. Analytical solution

Here, the modified Adomian decomposition method, proposed by Duan and Rach [15], is applied to obtain a solution for the governing differential equation of the nano-actuator, Eq. (1), subject to the boundary conditions, Eq. (2). Now, starting with the standard ADM, Eq. (1) is written in the operator form as:

$$Lu - \left(\eta \int_0^1 N_2 u dx + P\right)\frac{d^2u}{dx^2} = N_1 u, \tag{3}$$

where $L(.) = d^4/dx^4(.)$ is the linear differential operator to be inverted. N_1u and N_2u are the analytic nonlinear terms where

$$N_1 u = -\frac{\alpha}{u^\zeta} - \frac{\beta}{(\kappa+u)^2} - \frac{\gamma}{u}, \tag{4}$$

$$N_2 u = \left(\frac{du}{dx}\right)^2. \tag{5}$$

Attention to the boundary conditions shows that the function and its first derivative are given at $x = 0$ and $x = 1$. Hence, following the Duan–Rach ADM, the inverse operator (L^{-1}) is selected as:

$$L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x (.)dxdxdxdx. \tag{6}$$

Applying the inverse operator to both sides of Eq. (3) yields:

$$L^{-1}Lu-\left(\eta\int_0^1 N_2u dx+P\right)L^{-1}\frac{d^2u}{dx^2}=L^{-1}N_1u, \tag{7}$$

where the term of $L^{-1}L$ is evaluated by $L^{-1}(\frac{d^4u}{dx^4})=\int_0^x\int_0^x\int_0^x\int_0^x(u^{(4)})dxdxdxdx$ which results in:

$$L^{-1}Lu=u(x)-u(0)-xu'(0)-\frac{x^2}{2}u''(0)-\frac{x^3}{6}u'''(0), \tag{8}$$

where a prime denotes differentiation respect to the independent variable of x. From the boundary equation, Eq. (2), it is clear that $u(0)=1$ and $u'(0)=0$. Using the boundary conditions at $x=0$ and substituting Eq. (8) in Eq. (7) yields:

$$u(x)=1+\frac{x^2}{2}u''(0)+\frac{x^3}{6}u'''(0)+\left(\eta\int_0^1 N_2u(x)dx+P\right)\times L^{-1}u''(x)+L^{-1}N_1u(x), \tag{9}$$

where $u''(0)$ and $u'''(0)$ are undetermined coefficients. In the Wazwaz ADM method [19], which was utilized in previous studies (for example in [2]), the values of $u''(0)$ and $u'''(0)$ remained undetermined, and Eq. (9) started to be decomposed. In the regular ADM method adopted by pervious researchers, the integral term of $\int_0^1(\frac{du}{dx})^2dx$has been treated as an unknown coefficient (i.e. $T=\eta\int_0^1(\frac{du}{dx})^2dx$). Hence, these undetermined coefficients remain in the solution. Later, they will be determined using the boundary conditions at $x=1$, i.e. $u(1)=1$ and $u'(1)=0$. However, in the new method of Duan–Rach, these undetermined coefficients, *i.e.* $u''(0)$ and $u'''(0)$, should be evaluated using the remaining boundary conditions before decomposing the solution. It is worth noting that the integral term is also written in the operator form as N_2u. To achieve this purpose, Eq. (9) is evaluated at $x=1$:

$$u(1)=1+\frac{1}{2}u''(0)+\frac{1}{6}u'''(0)+\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[L^{-1}u''(x)\right]_{x=1}+\left[L^{-1}N_1u(x)\right]_{x=1}. \tag{10}$$

Differentiating Eq. (9) and evaluating at $x=1$ leads to:

$$u'(1)=u''(0)+\frac{1}{2}u'''(0)+\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[\frac{d}{dx}L^{-1}u''(x)\right]_{x=1}+\left[\frac{d}{dx}L^{-1}N_1u(x)\right]_{x=1}, \tag{11}$$

where $u(1)=1$ and $u'(1)=0$. Solving the set of Eqs. (10) and (11) for undetermined values of $u''(0)$ and $u'''(0)$:

$$\begin{aligned}u''(0)&=-6\left(\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[L^{-1}u''(x)\right]_{x=1}+\left[L^{-1}N_1u(x)\right]_{x=1}\right)\\&\quad+2\left(\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[\frac{d}{dx}L^{-1}u''(x)\right]_{x=1}+\left[\frac{d}{dx}L^{-1}N_1u(x)\right]_{x=1}\right)\\u'''(0)&=12\left(\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[L^{-1}u''(x)\right]_{x=1}+\left[L^{-1}N_1u(x)\right]_{x=1}\right)\\&\quad-6\left(\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[\frac{d}{dx}L^{-1}u''(x)\right]_{x=1}+\left[\frac{d}{dx}L^{-1}N_1u(x)\right]_{x=1}\right).\end{aligned} \tag{12}$$

Substituting the evaluated values of $u''(0)$ and $u'''(0)$ from Eq. (12) into Eq. (9) results:

$$\begin{aligned}u(x)&=1+\left(2x^3-3x^2\right)\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[L^{-1}u''(x)\right]_{x=1}+\left(2x^3-3x^2\right)\left[L^{-1}N_1u(x)\right]_{x=1}\\&\quad+\left(-x^3+x^2\right)\left(\eta\int_0^1 N_2u(x)dx+P\right)\times\left[\frac{d}{dx}L^{-1}u''(x)\right]_{x=1}+\left(-x^3+x^2\right)\left[\frac{d}{dx}L^{-1}N_1u(x)\right]_{x=1}\\&\quad+\left(\eta\int_0^1 N_2u(x)dx+P\right)\times L^{-1}u''(x)+L^{-1}N_1u(x).\end{aligned} \tag{13}$$

It is worth noticing that Eq. (13) is free of any undetermined coefficient. Now, the solution ($u(x)$) and the nonlinearities (N_1u and N_2u) in Eq. (13) are decomposed to:

$$u(x)=\sum_{m=0}^{\infty}u_m(x),\ N_1u=\sum_{m=0}^{\infty}A_m(x),\ N_2u=\sum_{m=0}^{\infty}B_m(x), \tag{14}$$

where $A_m=A_m(u_0,u_1,\ldots,u_m)$ and $B_m=B_m(u_0,u_1,\ldots,u_m)$ are the Adomian polynomials with the following definition which was first proposed by Adomian and Rach [20] as:

$$A_m=\frac{1}{m!}\frac{d^m}{d\lambda^m}N_1\left(\sum_{k=0}^{\infty}u_k\lambda^k\right)\Bigg|_{\lambda=0},\qquad m\ge 0, \tag{15}$$

$$B_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}N_2\left(\sum_{k=0}^{\infty}u_k\lambda^k\right)\Bigg|_{\lambda=0}, \qquad m \ge 0, \tag{16}$$

Substituting the Adomian polynomials in Eq. (13) and decomposing the solution leads to the following recursive scheme:

$$u_0(x) = c, \tag{17a}$$

$$\begin{aligned} u_1 = {} & 1 - c + \left(2x^3 - 3x^2\right)\left(\eta\int_0^1 B_0(x)dx + P\right) \times \left[L^{-1}u''_0(x)\right]_{x=1} + \left(2x^3 - 3x^2\right)\left[L^{-1}A_0(x)\right]_{x=1} \\ & + \left(-x^3 + x^2\right)\left(\eta\int_0^1 B_0(x)dx + P\right) \times \left[\frac{d}{dx}L^{-1}u''_0(x)\right]_{x=1} + \left(-x^3 + x^2\right)\left[\frac{d}{dx}L^{-1}A_0(x)\right]_{x=1} \\ & + \left(\eta\int_0^1 B_0(x)dx + P\right) \times L^{-1}u''_0(x) + L^{-1}A_0(x), \end{aligned} \tag{17b}$$

$$\begin{aligned} u_{m+1} = {} & \left(2x^3 - 3x^2\right)\left(\eta\int_0^1 B_m(x)dx + P\right) \times \left[L^{-1}u''_m(x)\right]_{x=1} + \left(2x^3 - 3x^2\right)\left[L^{-1}A_m(x)\right]_{x=1} \\ & + \left(-x^3 + x^2\right)\left(\eta\int_0^1 B_m(x)dx + P\right) \times \left[\frac{d}{dx}L^{-1}u''_m(x)\right]_{x=1} + \left(-x^3 + x^2\right)\left[\frac{d}{dx}L^{-1}A_m(x)\right]_{x=1} \\ & + \left(\eta\int_0^1 B_m(x)dx + P\right) \times L^{-1}u''_m(x) + L^{-1}A_m(x), \end{aligned} \tag{17c}$$

where c is the Duan's parameter, which accelerates the convergence of the solution [15,16].

The non-linearity of $Nu = f(u)$, in which f is solely a function of u, can be obtained as [21]:

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= f'(u_0)u_1, \\ A_2 &= f'(u_0)u_2 + f''(u_0)\frac{u_1^2}{2!}, \\ A_3 &= f'(u_0)u_3 + f''(u_0)u_1u_2 + f'''(u_0)\frac{u_1^3}{3!}. \end{aligned} \tag{18}$$

Using Eq. (15) or Eq. (18), the Adomian polynomials of the nonlinearity A_m are obtained as:

$$\begin{aligned} A_0 &= -\frac{\alpha}{u_0^4} - \frac{\beta}{(\kappa + u_0)^2} - \frac{\gamma}{u_0}, \\ A_1 &= u_1\left(\frac{4\alpha}{u_0^5} + \frac{2\beta}{(\kappa + u_0)^3} + \frac{\gamma}{u_0^2}\right), \\ A_2 &= u_2\left(\frac{4\alpha}{u_0^5} + \frac{2\beta}{(\kappa + u_o)^3} + \frac{\gamma}{u_0^2}\right) - \frac{1}{2}u_1^2\left(\frac{20\alpha}{u_0^6} + \frac{6\beta}{(\kappa + u_0)^4} + \frac{2\gamma}{u_0^3}\right), \\ A_3 &= u_3\left(\frac{4\alpha}{u_0^5} + \frac{2\beta}{(\kappa + u_o)^3} + \frac{\gamma}{u_0^2}\right) - u_1u_2\left(\frac{20\alpha}{u_0^6} + \frac{6\beta}{(\kappa + u_0)^4} + \frac{2\gamma}{u_0^3}\right) \\ & \quad + \frac{1}{6}\left(\frac{120\alpha}{u_0^7} + \frac{24\beta}{(\kappa + u_0)^5} + \frac{6\gamma}{u_0^4}\right)u_1^3. \end{aligned} \tag{19}$$

The non-linearity terms for B_m (i.e. $Nu = f(u')$) are obtained as:

$$\begin{aligned} B_0 &= u_0'^2, \\ B_1 &= 2u'_0u'_1, \\ B_2 &= u_1'^2 + 2u'_0u'_2, \\ B_3 &= 2u'_0u'_3 + 2u'_1u'_2. \end{aligned} \tag{20}$$

Following the recursive scheme (Eq. (17)) and using the Adomian polynomials, (Eqs. (19) and (20)), the first three stages of the Adomian solution are obtained as:

$$u_0 = c, \tag{21a}$$

Table 1
Maximum deflection of an actuator when $\alpha = 20$, $\zeta = 4$, $\beta = 5$, $\gamma = 0.325$, $\eta = 0.96$, $P = 0$ and $\kappa = -0.396$.

	ϕ_1 ($O(x^5)$)	ϕ_2($O(x^9)$)	ϕ_3($O(x^{13})$)	ϕ_4($O(x^{17})$)	ϕ_5($O(x^{21})$)
$c = 0.9$	0.13148	0.13263	0.13436	0.13508	0.13545
$c = 1.0$	0.10449	0.12591	0.13347	0.13674	0.13832
Numeric present	0.13534				
ADM [2]	–	–	–	0.13360	0.13486
Numerical [2]	0.13536				

$$u_1 = (1 - c) + \frac{1}{24}h_1x^2 - \frac{1}{12}h_1x^3 + \frac{1}{24}h_1x^4, \tag{21b}$$

$$u_2 = -\frac{1}{24}(x-1)^2x^2\left(\begin{array}{l} -\frac{1}{1680}h_2h_1x^4 + \frac{1}{840}h_2h_1x^3 - \frac{1}{30}\left(P - \frac{1}{168}h_2\right)h_1x^2 \\ +\frac{1}{30}h_1\left(P - \frac{1}{42}h_2\right)x + \left(c - \frac{1}{560}h_1 - 1\right)h_2 + \frac{1}{60}Ph_1 \end{array}\right), \tag{21c}$$

$$u_3 = \frac{1}{48}x^2(x-1)^2\left(h_5 + h_6x + h_7x^2 + h_8x^3 + h_9x^4 + h_{10}x^5 + h_{11}x^6 + h_{12}x^7 + h_{13}x^8\right), \tag{21d}$$

where $h_1 - h_{13}$ are shown in Appendix. It should be noticed that $h_1 - h_{13}$ are solely a function of the non-dimensional parameters, (i.e. α, ζ, β, γ, η, P, and κ) and Duan's parameter c, and they are not functions of the independent variable x; hence, $h_1 - h_{13}$ are only constant numbers for a specified actuator with the prescribed parameters (prescribed values of α, ζ, β, γ, η, P, and κ). Thus, the length of the approximate solution of ϕ_m for a specified case, i.e. prescribed values of non-dimensional parameters, reduces significantly. It should also be noticed that $\phi_m = \sum_{i=0}^{m} u_i(x)$. In the next section, the work of Yazdanpanahi et al. [2], which contains most of the non-dimensional parameters of Eq. (1), is adopted as good potential case study.

3. Case study

In the work of Yazdanpanahi et al. [2], the effect of the presence of a layer of water beneath of a nano-actuator is analyzed. The governing equation of the actuator is as follows:

$$\frac{d^4u}{dx^4} - \left(\eta\int_0^1\left(\frac{du}{dx}\right)^2dx\right)\frac{d^2u}{dx^2} = -\frac{\alpha}{u^4} - \frac{\beta}{(\kappa + u)^2} - \frac{\gamma}{u}, \tag{23}$$

where α, β, γ, η and κ denote the non-dimensional parameters of Casmir, applied voltage, fringing field effect, axial load, and the dielectric layer, respectively. The effect of the axial load of P was neglected in the work of Yazdanpanahi et al. [2], and thus $P = 0$. The boundary conditions are the same as those introduced in Eq. (2). Yazdanpanahi et al. [2] utilized the Wazwaz modified Adomian decomposition method [19] with undetermined coefficients to obtain an analytic solution for buckling of the nano-actuator. The authors compared the results of Wazwaz ADM with the results of a numerical method in a typical case when $\alpha = 20$, $\zeta = 4$, $\beta = 5$, $\gamma = 0.325$, $\eta = 0.96$, $\kappa = -0.396$. In this case, a comparison between the results of the present study and the results reported by Yazdanpanahi et al. [2] is performed in Table 1. The maximum deflection occurs at $x = 0.5$ because of the symmetry of the beam. When the beam is at rest free of any applied force, the shape of the beam is a straight line which can be represented by $u(x) = 1$. Applying any force results in the deflection of the beam into the substrate, and hence, $1 - u(x)$ denotes the actual deflection the beam from the rest position. Table 1 shows the values of $1 - u(0.5)$, which is the maximum deflection of the beam, as reported in [2].

In the present study, Eq. (1) subject to the boundary conditions of Eq. (2) is also solved numerically using a finite element code [22] with relative accuracy of 10^{-9}. The results of numerical solution is brought in Table 1 and compared with the numerical solution reported by [2]. The numerical method in the study of Yazdanpanahi et al. [2] was performed using the Runge Kutta method. In the Runge Kutta method, the size steps of the integrations was controlled using an automatic error check as described in [23,24] with the relative error below 10^{-10}.

As mentioned, Duan's parameter is a constant value which can increase the convergence rate the solution. The case of $c = 1.0$ discards the effect of Duan's parameter and reduces the solution to the Duan–Rach ADM solution without embedding the fast convergent parameter of c. As a comparison, the maximum deflection of the beam (i.e. $1 - u(0.5)$) for two different values of $c = 0.9$ and $c = 1$ and different stages of ADM are brought in Table 1. Table 1 shows that the presence of Duan's parameter remarkably increases the accuracy of the solution.

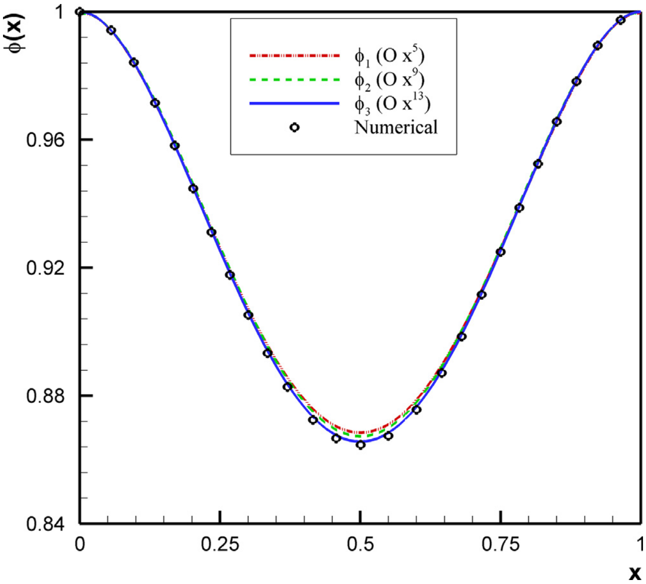

Fig. 1. The shape of the actuator evaluated using different stages of Duan–Rach ADM and comparison with numerical results ($\alpha = 20$, $\zeta = 4$, $\beta = 5$, $\gamma = 0.325$, $\eta = 0.96$, $P = 0$, $c = 0.9$ and $\kappa = -0.396$).

A comparison between the shapes of the nano-actuator evaluated by ϕ_1, ϕ_2 and ϕ_3 as well as the numerical solution is performed in Fig. 1. As seen, the increase of the stages of the Duan–Rach ADM solution increases the accuracy of the solution. It is clear that three stages of the solution ϕ_3, i.e. $\phi_3(\mathrm{O}(x^{13}))$, provides acceptable accuracy. The series of the solution (u_i and $i = 0..3$) for the actuator ($\alpha = 20$, $\zeta = 4$, $\beta = 5$, $\gamma = 0.325$, $\eta = 0.96$, $P = 0$, $\kappa = -0.396$ and $c = 0.9$) are calculated as:

$$u_0 = 0.9, \tag{24a}$$

$$u_1 = 0.1 - 2.1037113x^2 + 4.2074226x^3 - 2.1037113x^4, \tag{24b}$$

$$u_2 = 0.0876501x^2 - 0.5321969x^3 + 0.8906677x^4 + 1.0706901x^7 - 1.2491384x^6 - 0.2676725x^8, \tag{24c}$$

$$\begin{aligned} u_3 = {} & -0.0411973x^2 + 0.1530188x^3 - 0.0021534x^5 - 0.2528806x^4 + 0.7654223x^6 - 0.7448388x^7 \\ & -1.3386473x^8 + 3.5623600x^9 - 3.2582619x^{10} + 1.3886126x^{11} - 0.2314354x^{12}, \end{aligned} \tag{24d}$$

where $\phi_m = u_0 + u_1 + \cdots + u_m$. The series of Eq. (24) were utilized for calculation of approximate solutions of ϕ_m.

In the study of Yazdanpanahi et al. [2], utilizing the Wazwaz-ADM method with undetermined coefficients, the recursive scheme containing undetermined parameters [2] were written as:

$$u_0 = 1, \tag{25a}$$

$$u_1 = C_2\frac{x^2}{2} + C_3\frac{x^3}{6} + T\int_0^x\int_0^x u_0 dxdx + L^{-1}A_0(x), \tag{25b}$$

$$u_{m+1} = T\int_0^x\int_0^x u_m dxdx + L^{-1}A_m(x), \tag{25c}$$

where C_2, C_3 and T are undetermined coefficients which should be evaluated using $T = \eta\int_0^1 (\frac{du}{dx})^2 dx$ and the boundary conditions at $x = 1$ (i.e. $u(1) = 1$ and $u'(1) = 0$). For example, in the studied case of nano-actuator ($\alpha = 20$, $\zeta = 4$, $\beta = 5$, $\gamma = 0.325$, $\eta = 0.96$, $P = 0$ and $\kappa = -0.396$) and ϕ_3, the method of Wazwaz-ADM with undetermined coefficients results in:

$$\begin{aligned} \phi_3 = {} & 1 - 0.0539631x^{12} + 0.0190414C_3x^{11} \\ & +(-0.0017232C_3^2 + 0.089897T + 0.0922506C_2)x^{10} + \\ & +(-0.01723201C_3C_2 - 0.0165398TC_3)x^9 \\ & +(-0.0868230C_2T - 0.04113997T^2 - 0.04652644C_2{}^2 - 0.1059331)x^8 \end{aligned}$$

$$
\begin{aligned}
&+(0.0001984T^2C_3+0.0249197C_3)x^7\\
&+(0.1744376C_2+0.0013889T^2C_2+0.0013889T^3+0.1272044T)x^6\\
&+0.0083333TC_3x^5+(0.0416667T^2+0.0416668C_2T-1.4169948)x^4\\
&+0.1666667C_3x^3+(0.5000000C_2+0.5000000T)x^2.
\end{aligned}
\tag{26}
$$

In order to determine the values of C_2, C_3 and T, the following nonlinear algebraic equations should be solved simultaneously:

form $u(1)=1$:

$$
\begin{aligned}
&0.00138889T^3+(0.0005267+0.0001984127C_3+0.0013889C_2)T^2\\
&+(-0.0451563C_2+0.7171009-0.0082065C_3)T-0.0465264{C_2}^2\\
&+(0.7666881-0.0172320C_3)C_2-1.5768910+0.2106277C_3-0.0017232{C_3}^2=0,
\end{aligned}
\tag{27a}
$$

from $u'(1)=0$:

$$
\begin{aligned}
&0.0083333T^3+(-0.1624531+0.0013889C_3+0.0083333C_2)T^2\\
&+(-0.5279170C_2+2.6621914-0.1071915C_3)T-0.3722115{C_2}^2\\
&+(2.9691310-0.1550881C_3)C_2+0.8838930C_3-0.0172320{C_3}^2-7.1630014=0,
\end{aligned}
\tag{27b}
$$

and from $T=\eta\int_0^1\left(\frac{du}{dx}\right)^2dx$:

$$
\begin{aligned}
&-0.0000061T^6\\
&+(-0.0000121C_2+0.0001088-0.0000019C_3)T^5\\
&+\left(-0.0000061{C_2}^2+(-0.0000019C_3+0.0006674)C_2+0.0001217C_3-0.0055224-1.424501410^{(-7)}{C_3}^2\right)T^4\\
&+\left(0.0010167{C_2}^2+(0.0003686C_3-0.0162464)C_2+0.0000347{C_3}^2+0.0391907-0.0034769C_3\right)T^3\\
&\begin{pmatrix}0.0004581{C_2}^3+(0.0002481C_3-0.0241244){C_2}^2+(0.0000460{C_3}^2-0.0096250C_3\\+0.2269145)C_2-0.9510121+0.0000027{C_3}^3-0.0009536{C_3}^2+0.0454145C_3\end{pmatrix}T^2\\
&\begin{pmatrix}-0.0222641{C_2}^3+(0.3631448-0.0129583C_3){C_2}^2+(-2.3985198+0.1602775C_3-0.0025806{C_3}^2)C_2\\-0.6500819C_3+5.4189587+0.0176223{C_3}^2-0.0001752{C_3}^3\end{pmatrix}T\\
&-0.0088667{C_2}^4+(-0.0069271C_3+0.1757213){C_2}^3+(-0.0020827{C_3}^2-1.4632480+0.1202512C_3){C_2}^2\\
&+(-0.0002851{C_3}^3-0.7742099C_3+4.8411871+0.0272031{C_3}^2)C_2\\
&-0.0000150\left({C_3}^2-17.5494349C_3+78.4127108\right)\left({C_3}^2-121.4701752C_3+4958.097920\right).
\end{aligned}
\tag{27c}
$$

Solving Eqs. (27a)–(27c) demands numerical procedures. In addition, finding and choosing an appropriate set of roots for C_2, C_3 and T is a challenge. In contrast with the method of undetermined coefficients, the Duan–Rach-ADM method explicitly gives the shape of the actuator free of any undetermined coefficient. In addition, the parametric analysis of the variables is also possible using the Duan–Rach Adomian method. For instance, assume an actuator with the prescribed values of $\alpha=20$, $\zeta=4$, $\gamma=0.325$, $\eta=0.96$, $P=0$ and $\kappa=-0.396$ and adjustable value of voltage parameter of β. Now, suppose that an engineer needs 0.15 deflection of actuator (i.e. $1-u(0.5)=0.15$) in an application of the mentioned nano-actuator with the given prescribed non-dimensional parameter. Therefore, the required voltage is the unknown of the problem. In this case, as an example, the maximum deflection as a function of the voltage parameter β, is easily obtained as:

$$
\phi_3(0.5)=-0.0000183b^3-0.0004542b^2+0.9349773-0.0111415b,
\tag{28}
$$

by using Duan–Rach ADM and ϕ_3.

Solving for $\phi_3(0.5)=0.85$ results:

$$
\{\mathrm{b}=5.88\},\{\mathrm{b}=-15.38+23.55\mathrm{i}\},\{\mathrm{b}=-15.38-23.55\mathrm{i}\},
\tag{29}
$$

where solely $b=5.88$ is a real acceptable solution. It is worth noticing that obtaining such results using the ADM with undetermined coefficients leads to solution of four highly nonlinear algebraic equations; two for C_2 and C_3, one for T and another one for β. The new ADM facilities advanced practical relations for behavior of actuator as a simultaneous function of non-dimensional parameters. For example, the following explicit relation, using ϕ_3 and $\alpha=20$, $\zeta=4$, $\gamma=0.325$, $P=0$ and $\beta=5$, is obtained for the maximum deflection of the beam as a function of η and κ:

$$
\begin{aligned}
\phi_3(0.5)=\;&(0.9349873-0.0000104\eta)\kappa^9+(7.5733970-0.0000843\eta)\kappa^8\\
&+(27.2472940-0.0003041\eta)\kappa^7+(57.1497197-0.0006416\eta)\kappa^6\\
&+(-0.0008733\eta+77.0141041)\kappa^5+(69.1499535-0.0007949\eta)\kappa^4\\
&+(41.3693531-0.0004833\eta)\kappa^3+(-0.0001889\eta+15.9011514)\kappa^2\\
&+(3.5630807-0.0000428\eta)\kappa+0.354598405-0.0000043\eta.
\end{aligned}
$$

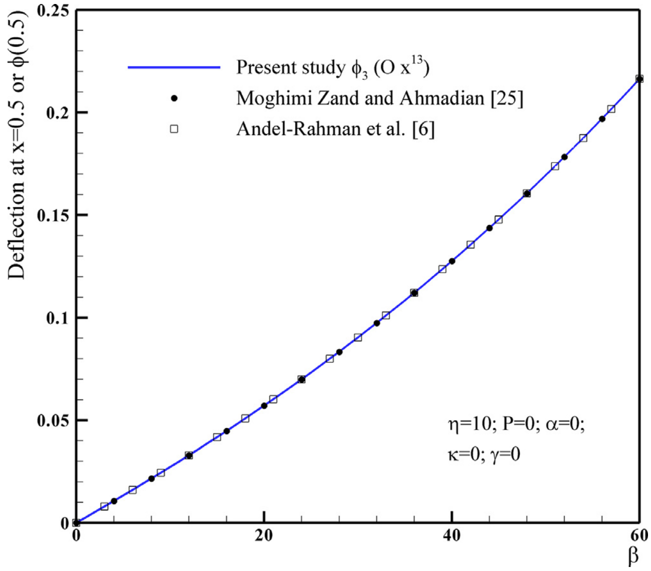

Fig. 2. A comparison between the results of the present study and the finite-element results reported by Moghimi-Zand and Ahmadian [25] and Abdel-Rahman et al. [6] for a nano-switch subject to electrostatic force.

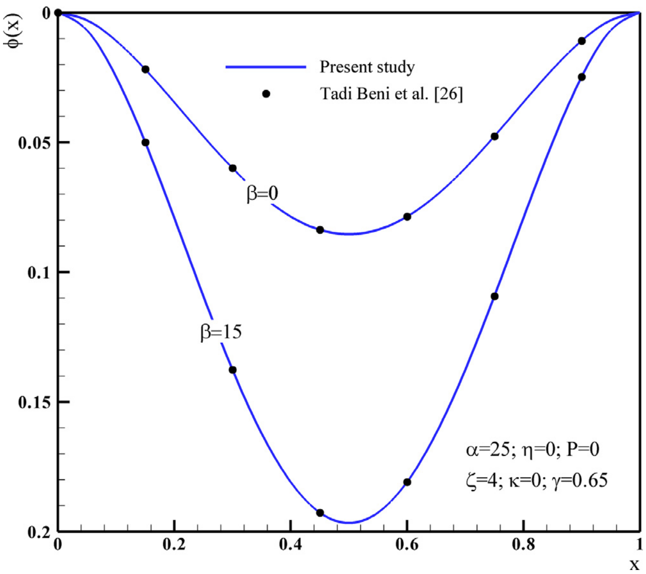

Fig. 3. A comparison between the finite-element numerical results reported by Tadi-Beni et al. [26] and the results of the present analytical solution.

Fig. 2 depicts a comparison between the results of the present study (Duan-Rach with $c = 0.9$) and the finite-element results reported by Moghimi-Zand and Ahmadian [25] and Abdel-Rahman et al. [6] for a nano-switch subject to electrostatic force. Fig. 2 shows the displacement of the mid-point position of the nano-switch (i.e. $u(0.5)$) as a function of applied voltage (β). As seen, the results of the present analytical solution are in good agreement with the numerical results available in literature.

Fig. 3 shows a comparison between the finite-element numerical results reported by Tadi-Beni et al. [26] and the results of the present analytical solution when $c = 0.9$. This figure illustrates the calculated shape of a nano-switch subject to the Casimir effect ($\alpha = 25$) and electrostatic forces ($\beta = 0$ and $\beta = 15$). Fig. 3 shows that the present analytical solution is in good agreement with the results reported by Tadi-Beni et al. [26].

4. Conclusion

The governing equation of clamped-clamped nano-actuators is considered in a general form, in which the nano-actuators are subject to different nonlinear forces such as the van der Waals force, Casimir force, applied voltage, fringing field effect, capillary effect and the dielectric layer effect. In addition, there is a nonlinear integral term in the governing equation of the nano-actuator because of the presence of the axial loads. The Duan–Rach Adomian Decomposition Method (ADM) is successfully applied to obtain a new analytic solution for the buckling of nano-actuators. The obtained solution is free of any undetermined coefficient. An acceleration parameter, Duan's parameter, is embedded in the solution to increase the convergence rate of the Adomian series. As a case study, the results are compared with the results of the conventional ADM with undetermined coefficients as well as the results of an accurate numerical solution. The result indicates that using more stages of the Adomian solution increases the accuracy of the solution. However, only using three stages of the Duan–Rach ADM provides the results, which is sufficient for engineering applications. Furthermore, the presence of Duan's parameter, c, remarkably increases the convergence rate of the solution. The present solution provides relations, describing the behavior of the nano-actuator as a function of the non-dimensional parameters. Such relations can be conveniently utilized in the design and application of nano-actuators.

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Appendix

$$h_1 = -\frac{\alpha}{c^\zeta} - \frac{\beta}{(\kappa + c)^2} - \frac{\gamma}{c}, \tag{h1}$$

$$h_2 = +\frac{\alpha\zeta}{c^\zeta c} + \frac{2\beta}{(\kappa + c)^3} + \frac{\gamma}{c^2}, \tag{h2}$$

$$h_3 = -\frac{1}{24}\frac{\alpha\zeta}{c^\zeta c} - \frac{1}{12}\frac{\beta}{(\kappa + c)^3} - \frac{1}{24}\frac{\gamma}{c^2}, \tag{h3}$$

$$h_4 = -\frac{\alpha\zeta^2}{c^\zeta c^2} - \frac{\alpha\zeta}{c^\zeta c^2} - \frac{6\beta}{(\kappa + c)^4} - \frac{2\gamma}{c^3}, \tag{h4}$$

$$\begin{aligned} h_5 = {} & \frac{1}{38102400}\eta h_1^3\left(P - \frac{1}{12}h_2\right) + h_1^2\left(\frac{1}{907200}\eta(c-1)h_2 + \frac{1}{266112}h_4\right) \\ & + h_1\left(h_2\left(-\frac{1}{15120}P - \frac{71}{415800}h_3\right) + \frac{1}{1260}P^2 + \frac{13}{6300}Ph_3 - \frac{1}{280}h_4(c-1)\right) \\ & + (c-1)\left(h_2\left(\frac{3}{35}h_3 + \frac{1}{30}P\right) + h_4(c-1)\right), \end{aligned} \tag{h5}$$

$$\begin{aligned} h_6 = {} & \frac{1}{19051200}\eta\left(P - \frac{1}{12}h_2\right)h_1^3 + \left(\frac{1}{453600}\eta(c-1)h_2 + \frac{1}{498960}h_4\right)h_1^2 \\ & + h_1\left(\left(-\frac{1}{7560}P - \frac{4}{51975}h_3\right)h_2 + \frac{1}{630}P^2 + \frac{1}{1050}Ph_3 - \frac{1}{630}h_4(c-1)\right) \\ & + \frac{1}{15}h_2(c-1)\left(P + \frac{4}{7}h_3\right), \end{aligned} \tag{h6}$$

$$\begin{aligned} h_7 = {} & -\frac{1}{76204800}\eta h_1^3\left(-\frac{1}{4}h_2 + P\right) + h_1^2\left(-\frac{1}{453600}\eta(c-1)h_2 + \frac{1}{3991680}h_4\right) \\ & + h_1\left(h_2\left(\frac{1}{59400}h_3 + \frac{1}{10080}P\right) - \frac{1}{2520}P^2 - \frac{1}{6300}Ph_3 + \frac{1}{2520}h_4(c-1)\right) \\ & - \frac{1}{15}h_2(c-1)\left(\frac{1}{7}h_3 + P\right), \end{aligned} \tag{h7}$$

$$\begin{aligned} h_8 = {} & -\frac{1}{12700800}\eta h_1^3\left(-\frac{1}{45}h_2 + P\right) - \frac{1}{665280}h_4h_1^2 - \frac{2}{35}h_3h_2(c-1) \\ & + h_1\left(h_2\left(\frac{1}{18900}P + \frac{23}{207900}h_3\right) - \frac{1}{420}P^2 - \frac{2}{1575}Ph_3 + \frac{1}{420}h_4(c-1)\right), \end{aligned} \tag{h8}$$

$$h_9 = \frac{1}{25401600}\eta h_1^3\left(\frac{1}{180}h_2 + P\right) - \frac{13}{3991680}h_4 h_1^2 + \frac{1}{35}h_3 h_2(c-1)$$
$$+h_1\left(h_2\left(-\frac{1}{29700}h_3 + \frac{1}{151200}P\right) + \frac{1}{840}P^2 - \frac{1}{6300}Ph_3 - \frac{1}{840}h_4(c-1)\right), \quad \text{(h9)}$$

$$h_{10} = -\frac{1}{762048000}\left(\eta h_1^2 h_2 + \frac{42000}{11}h_4 h_1 + h_2\left(\frac{161280}{11}h_3 + 30240P\right) - 725760Ph_3\right), \quad \text{(h10)}$$

$$h_{11} = \frac{1}{2286144000}h_1\left(\eta h_1^2 h_2 + \frac{453600}{11}h_4 h_1 + h_2\left(-\frac{120960}{11}h_3 + 30240P\right) - 725760Ph_3\right), \quad \text{(h11)}$$

$$h_{12} = -\frac{1}{71280}h_1\left(-\frac{24}{35}h_3 h_2 + h_4 h_1\right), \quad \text{(h12)}$$

$$h_{13} = \frac{1}{285120}h_1\left(-\frac{24}{35}h_3 h_2 + h_4 h_1\right). \quad \text{(h13)}$$

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